The Weak-Coupling Limit of $3D$ Simplicial Quantum Gravity

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We investigate the weak-coupling limit, $\kappa \to \infty$, of $3D$ simplicial gravity using Monte Carlo simulations and a Strong Coupling Expansion. With a suitable modification of the measure we observe a transition from a branched polymer to a crinkled phase. However, the intrinsic geometry of the latter appears similar to that of non-generic branched polymer, probable excluding the existence of a sensible continuum limit in this phase.

1. INTRODUCTION

D–dimensional simplicial quantum gravity is a discretization of Euclidean quantum gravity with the integration over space-time metrics replaced by a sum over all possible D–dimensional triangulations constructed by gluing together equilateral simplexes. It is defined by the partition function

$$Z(\mu,\kappa) \;=\; \sum_{N_D} e^{-\mu N_D}\; Z(\kappa, N_D) \qquad (1)$$
$$\;=\; \sum_{N_D,N_0} e^{-\mu N_D + \kappa N_0}\; W_D(N_0,N_D)\;,$$

$$W_D(N_0,N_D) \;=\; \sum_{T\in\mathcal{T}(N_0,N_D)} \frac{1}{C_T}\;, \qquad (2)$$

where $N_D = \#$ D–simplexes, $N_0 = \#$ vertices, and C_T is the *symmetry* factor of a labeled triangulation T chosen from a suitable ensemble $\mathcal{T}$ (*eg* combinatorial). μ and κ are the discrete cosmological and Newton's coupling constants.

In $D = 3$ and 4 this model has *two* phases:

- $\kappa < \kappa_c$: a (intrinsically) *crumpled* phase
- $\kappa > \kappa_c$: a *branched polymer* phase

separated (regrettably) by a *discontinuous* phase transition [1].

As a discontinuous phase transition excludes a sensible continuum limit, there have been several attempts to modify the model Eq. (1) in the hope of finding a non-trivial phase structure. This includes: adding a *measure* term [2]

$$W_D(N_0,N_D,\beta) \;=\; \sum_{T\in\mathcal{T}(N_0,N_D)} \frac{1}{C_T}\prod_{i=0}^{N_0} q_i^{\,\beta}, \qquad (3)$$

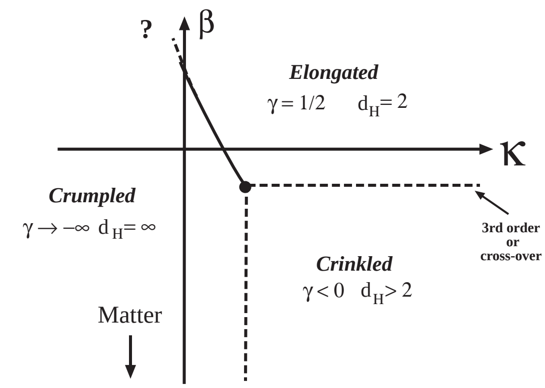

Figure 1. A schematic phase diagram of simplicial gravity in 3 and 4 dimensions.

where q_i is the order of the vertex i (number of simplexes containing i), and coupling matter fields to the geometry [3].

Such modifications do indeed lead to a more complicated phase diagram (Fig. 1), and suitable modified the model exhibits a new *crinkled* phase. But does this new phase structure imply a more interesting non-trivial critical behavior? To investigate this we have studied the weak-coupling limit, $\kappa \to \infty$, of the model Eq. (1) for $D = 3$.

2. THE EXTREMAL ENSEMBLE

In the weak-coupling limit the partition function Eq. (1), in $D = 3$ and 4, is expected to be dominated by an Extremal Ensemble (EE) of triangulations. For this ensemble, defined as tri-

angulations with the maximal ratio N_0/N_D, the partition function simplifies:

$$Z(\mu) \;=\; \sum_{N_D} \mathrm{e}^{-\mu N_D}\, W_D(N_D) \tag{4}$$

$$W_D(N_D) \;=\; \sum_{T\in\mathcal{T}(N_0^{\max},N_D)} \frac{1}{C_T}$$

where

$$N_0^{\max} \;=\; \begin{cases} \left\lfloor \dfrac{N_3+10}{3} \right\rfloor & D=3\,, \\ \left\lfloor \dfrac{N_4+18}{4} \right\rfloor & D=4\,. \end{cases} \tag{5}$$

Here $\lfloor x \rfloor$ denotes the floor function — the biggest integer not greater than x. This in turn defines several *distinct* series for the EE:

$$3D \;:\; S^0\,\big(N_0,\,3N_0{-}10\big),\;\; S^1\,\big(N_0,\,3N_0{-}9\big),\;\; S^2\,\big(N_0,\,3N_0{-}8\big).$$

$$4D \;:\; S^0\,\big(N_0,\,4N_0{-}18\big),\;\; S^1\,\big(N_0,\,4N_0{-}17\big).$$

Assuming the asymptotic behavior $W_D(N_D) \sim \exp(-\mu_c N_D)\, N_D^{\gamma-3}$, which defines the string susceptibility exponent γ, we observe (from a SCE) that for the different series S^k, $\gamma^k = k + \frac{1}{2}$. This difference in the exponent γ can be understood as the 'higher' series ($k = 1, 2, \ldots$) can be constructed by introducing k "defects" (marked points) into triangulations belonging to the minimal series S^0. Moreover, we observe that the minimal series appears to have very small finite-size effects.

The minimal series S^0 can be explicitly *enumerated* as it corresponds to D–dimensional *combinatorial stacked spheres* (CSS), *ie* to the surface of a $(D+1)$–dimensional simplicial cluster. The number of $(D+1)$–dimensional simplicial clusters build out of n $(D+1)$ simplexes, rooted at a marked outer face, is given by (where $n = N_0 - D - 1$) [4]

$$e_{D+1,n} \;=\; \sum_{\substack{n_1+\cdots+n_{D+1} \\ = n-1}} e_{D+1,n_1}\cdots e_{D+1,n_{D+1}}$$

$$=\; \frac{1}{nD+1}\begin{pmatrix} (D{+}1)\,n \\ n \end{pmatrix} \tag{6}$$

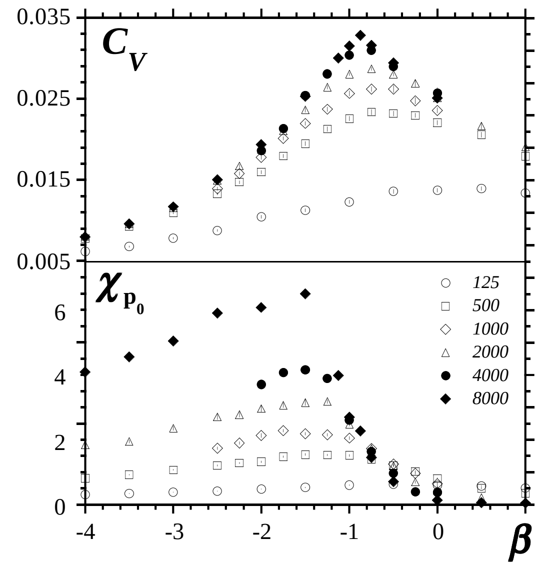

Figure 2. Evidence of a phase transition in the $3D$ EE, Eq. (4), with a modified measure Eq. (3) (TOP) The fluctuations in the measure term: C_V, and (BOTTOM) in the maximal vertex order: χ_{p_0}.

$$\Rightarrow \quad W_D(N_D) \;=\; \frac{D+2}{N_D}\, e_{D+1,\frac{N_D-2}{D}}\;. \tag{7}$$

Expanding this gives

$$W_3(N_3) = \frac{10}{\sqrt{2\pi}\,N_3^{5/2}}\left(\frac{256}{27}\right)^{\frac{N_3-2}{3}}\left(1+\frac{83}{48}\frac{1}{N_3}\cdots\right.$$

$$W_4(N_4) = \frac{6\sqrt{5}}{\sqrt{2\pi}\,N_4^{5/2}}\left(\frac{3125}{256}\right)^{\frac{N_4-2}{4}}\left(1+\frac{33}{20}\frac{1}{N_4}\cdots\right.$$

with $\gamma = 1/2$ as expected for branched polymers.

3. A MODIFIED MEASURE

We have investigated the $3D$ EE including a measure term, using both MC simulations and a SCE [5]. We find a *continuous* phase transition to a crinkled phase at $\beta \approx -1$ (Fig. 2). This is evident in the fluctuations both in the measure term — the "specific heat" C_V — and in the maximal vertex order p_0. Scaling analysis of the peak value of specific heat gives: $C_V^{\max} \approx a + bN_3^{-0.34(4)}$.

To explore the fractal properties of the geometry in the crinkled phase we have measured

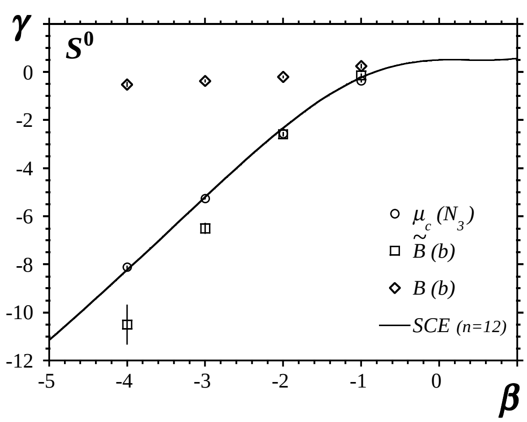

Figure 3. Variations in γ with β for the EE Eq. (4) with a modified measure Eq. (3).

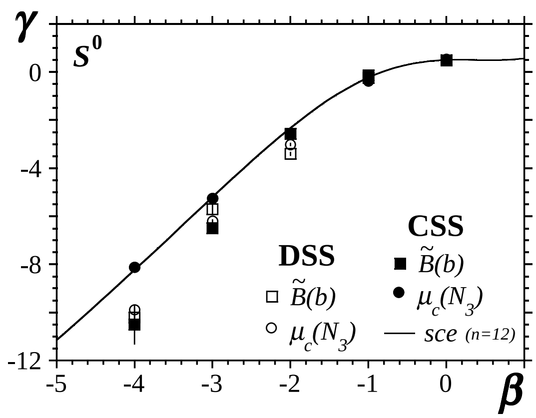

Figure 4. belfig4Variations in γ with a modified measure, both for an ensemble of DSS and CSS.

the variations in γ with β using several different methods (Fig. 3). As in $D = 4$, we find that γ becomes negative at β_c and decreases with β. Similarly we find a spectral dimension that increase from $d_s = 4/3$ for $\beta > \beta_c$, to $d_s \approx 2$ as $\beta \to \infty$.

Estimates of the intrinsic fractal dimension d_H differ, on the other hand, substantially depending on how it is defined — on the direct graph (from a vertex-vertex distribution) or on the dual graph (simplex-simplex distribution). The former yields $d_H \to \infty$, the latter $d_H \approx 2$. In addition, we observe that the crinkled phase appears dominated by a *gas* of sub-singular vertices.

Combined this evidence suggests that the crinkled phase probably corresponds to some kind of non-generic branched polymers phase which makes it unlikely that any sensible continuum limit exist in this phase. This of course does not exclude the possibility that a second phase order transition point exists somewhere on the phase boudary, for example at the end of the first order transition line (Fig. 1).

4. DEGENERATE TRIANGULATIONS

The EE can also be defined with the ensemble of *degenerate* triangulations introduced in Ref. [6]. In this case degenerate stacked spheres (DSS) are constructed by slicing open a face and inserting a vertex. Different from CSS, Eq. (5), DSS are defined by the maximal ratio:

$$\frac{N_0}{N_D} = \frac{1}{2} + \frac{D}{N_D}. \tag{8}$$

This ensemble can also be enumerated explicitly.

Modifying the measure leads to identical phase structure as is observed for CSS. This is shown in Fig. 4 where we plot the variations in γ with β for the two ensembles. That the two, very different, ensembles agree on the fractal structure is reassuring and reflects the universal properties of the crinkled phase.

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